MST204. GMM

The Open University

Mathematics/ Science/ Technology

An Inter-faculty Second Level Course

Mathematical Models and Methods

mathematical models and methods

Guide to mathematical modelling

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1 Introduction

This course is about the use of mathematics to solve problems in the real world. So far, with the exception of *Unit M*, the 'modelling' units have presented 'standard models' for describing physical situations; for instance, Newton's second law is a model describing the motion of a particle, and the logistic equation is a model of population growth.

Newton's second law is described in *Unit 4*; the logistic equation is described in *Unit 3*.

The process of mathematical modelling consists of three main stages. First, we take a problem set in the real world and *formulate* it as a mathematical problem. This mathematical problem is then *solved*, and finally the solution is translated back into the original context so that the results produced by the model can be *interpreted* and used to help solve the real problem.

You have seen this process at work in *Unit M*, and in the next few weeks you will have the chance to carry it out yourself in full.

The aims of this activity are summarized below.

1. To provide you with practice in modelling skills, such as problem formulation, interpretation and validation of models, and report writing.

2. To give you an opportunity to see modelling as a whole rather than a collection of parts.

3. To give you experience (at Summer School) of working in a group on a modelling problem, and to help you develop the skills of collective problem solving.

4. To help you acquire a better understanding of the relation between a mathematical model and the real world, and of which aspects of the real world can be effectively represented by a mathematical model.

5. To give you the experience of working on a modelling problem over a time-scale that is more realistic than are the times usually associated with examination and assignment questions.

6. To give you time to test your model against reality and improve it until you have produced a solution you feel to be satisfactory.

7. To give you the opportunity to write an account of your own modelling activities and to stand back from, analyse and critically comment on a model you have developed.

8. To encourage you to improve your mathematical knowledge or technique.

The mathematical modelling exercise you are about to tackle is organized as follows. While you are at Summer School you will spend a fair amount of time working on a mathematical modelling problem, in a group with fellow students, and with guidance from a tutor. When you get back from Summer School you will extend and complete your work on the problem, and you will write up your results in the form of a report to your course tutor. This report will make up the whole of TMA 07.

The modelling problems, from which you must choose just **one** to do, are given in the form of *Problem Specifications for the Modelling Exercise* in the current Assignment Booklet. Each specification consists essentially of a problem of non-mathematical origin, and your task is to use mathematical modelling to help you to solve this problem.

Three weeks are set aside in the Study Calendar, in addition to the time at Summer School, for this activity. The first of these weeks is just before the Summer Schools period, the other two immediately after it. (If you attend the last week of Summer School, you may find that your Summer School attendance cuts into the beginning of the second modelling week, because the study week for the course does not start on Sunday. If so, you would be well advised to get fully up-to-date with your study of the course units, or even ahead of the calendar if possible, before Summer School, in order to free some more time after Summer School to complete your modelling work.)

During the week set aside for modelling work before Summer School, you will be expected to decide which of the specified modelling problems you wish to tackle, and to do some preliminary work on it. You will then be in a good position to make a flying start on the modelling exercise when you get to Summer School. If you work effectively while you are at Summer School, you should have a well-developed model by the time you return. You will then have one week to check your results, compare your predictions with reality, criticize your model and make some improvements to it. You should have plenty of time to write your report in the second post Summer School modelling week.

This *Guide to Mathematical Modelling* is not like a conventional unit but is a resource to help you at different stages in your modelling work. You are not expected to read it through and study it from cover to cover. Rather, skim through it first, and then use the relevant sections before, during and after Summer School.

Before you choose a problem specification, therefore, you are advised to skim through this guide quickly to familiarize yourself with its contents. Then return to Section 2, which should help you in your choice of problem. In the course of deciding on the problem, you will carry out the preliminary work you are asked to do before Summer School.

Your work at Summer School will be supervised by the tutorial staff, so you should not need much help from this guide for that part of the exercise. Nevertheless, we have included a brief outline of the steps one has to go through in developing a mathematical model, in Section 3. You may find it helpful to read this just before you attend Summer School (and it should certainly be useful if you are unable to attend Summer School).

In Section 4 we describe the work you will do after Summer School, explain what is expected of you for TMA07, and give you some advice on writing your report.

There are two television programmes associated with this component of the course. One will be broadcast in the week before Summer Schools begin, and the other in the week after Summer Schools end. They are thematically linked to the two programmes associated with *Unit M*. Like those programmes, they are designed to complement your modelling activities; neither requires any preliminary or follow-up work, but each assumes that you are following the timetable for the modelling exercise which has been described above.

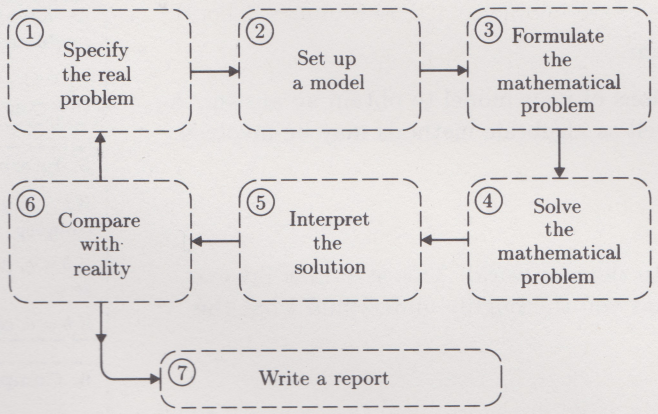
You may find it helpful during your modelling work to find one or two (non-mathematical) friends who might be interested in the problem that you are going to solve. Persuade them to be your *audience* and ask them to be willing to read your TMA07 when first drafted. Your audience will be useful for discussion when you get stuck and will help you to produce a report (TMA07) which is clear and easy to understand.

2 Before Summer School

The two main tasks to be carried out during the week before Summer School are renewing familiarity with the approach to modelling used in *Unit M*, and choosing a problem. It is important that you select a problem which interests you, and that you have some ideas about how to tackle it. We recommend that before you choose a problem you work through Subsections 2.1 to 2.3, which will give you useful background and practice in mathematical modelling. Then fill in the problem choice table in Subsection 2.4.

2.1 A model of modelling

In this guide we shall use the analysis and vocabulary for modelling developed in *Unit M*, and in particular we shall make use of the seven-box modelling diagram from Section 4. However, there is no single generally accepted psychological or mathematical account of the stages or activities people go through when modelling or problem-solving, so you should not be too surprised if the seven-box diagram below does not apply perfectly to your modelling experience or if you encounter different breakdowns of the modelling activities in your reading. Nevertheless, the diagram is a useful way of summarizing the various activities involves in mathematical modelling.


1 Specify the real problem
2 Set up a model
3 Formulate the mathematical problem
4 Solve the mathematical problem
5 Interpret the solution
6 Compare with reality
7 Write a report

The seven-box diagram

The left-hand column of the diagram represents the real world in which the specified problem originally arose. The right-hand column represents the ideal 'mathematical' world of symbols and equations. The middle column represents the translation process through which these two worlds meet, where one formalizes and simplifies the real problem to make it amenable to mathematical reasoning, and afterwards tries to assess the effect of these formalizations and simplifications on the solution obtained. Finally, the wide box at the bottom represents your activity at the conclusion of the modelling process. Very often this will be to explain to someone else what you have done. In some ways this box is the most important, for the ability to communicate intelligibly and avoid misunderstandings is important even in walks of life where mathematics has no part.

The boxes are numbered, and joined by arrows, to indicate the normal direction of travel from one to another, but it is usual to return to some of the boxes several times as one's ideas develop. In particular, it is frequently necessary to return to Box 1 after reaching Box 6, and go round the 'loop' a second or even a third time if the first attempt does not produce a satisfactory solution of the original problem. One can regard mathematical modelling as an iterative process, starting from a rather crude model and gradually improving it until finally it is good enough to provide an acceptable answer. This is a much more effective way of tackling modelling problems than trying to get everything right the first time round.

To illustrate the activities of each box in the seven-box diagram, here is a summary of the exponential model of the growth of a population given in Section 1 of *Unit 3*.

Box 1: Specify the real problem

The real situation which you are thinking about will have many aspects. Initially, these will be jumbled and confused in your mind. As you think about and explore the situation, the problem you wish to model mathematically should crystallize in your mind.

Box 2: Set up a model

In order to get to grips with the problem, it is often helpful to write down a list of all the features you can think of which may be relevant, and then separately select a few to work on. A mathematical model is a set of relations between variables representing these features, which in some sense describes the real situation. First, you must decide how the system behaves, and then you must express this mathematically. This step is creative and hard, unless you have met an appropriate model previously.

Box 3: Formulate the mathematical problem

Your mathematical model is a description of at least part of the real system. You must next decide what question you want to ask of the model. Usually the explicit behaviour of one or more of the variables is required.

Box 4: Solve the mathematical problem

This is a familiar step. You solve the equations of your model to obtain an answer to the mathematical question. Numerical as well as algebraic methods may be involved. The mathematics should be checked.

Box 5: Interpret the solution

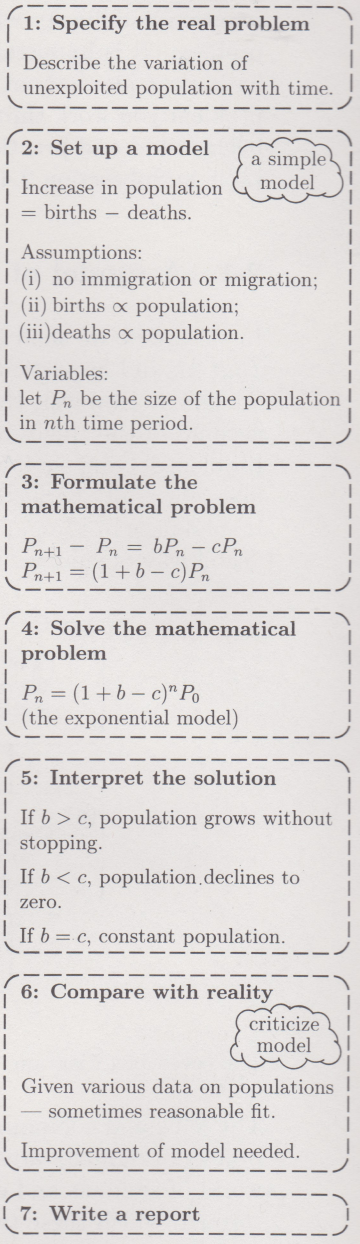
You now translate back from mathematics to the real world. This step may involve calculating values and plotting graphs so that you thoroughly understand what the model predicts.

Box 6: Compare with reality

You must now evaluate your model by comparing its predictions with the real world, collecting appropriate data and seeing how far the model succeeds in doing what you ask of it. Normally it will need improvement; if so, you should go back to Box 1 and go round the loop again. But at some point the model will be judged to be good enough for the purpose for which it is intended, and you can go on to Box 7.

Box 7: Write a report

This involves organizing and clearly presenting what you have found out, in a form which is appropriate to whoever will use your results.


1: Specify the real problem
Describe the variation of unexploited population with time.
2: Set up a model
a simple model
Increase in population = births − deaths.
Assumptions:
(i) no immigration or migration;
(ii) births ∝ population;
(iii)deaths ∝ population.
Variables:
let P_n be the size of the population in nth time period.
3: Formulate the mathematical problem
$P_{n+1} - P_n = bP_n - cP_n$
$P_{n+1} = (1 + b - c)P_n$
4: Solve the mathematical problem
$P_n = (1 + b - c)^n P_0$
(the exponential model)
5: Interpret the solution
If $b > c$, population grows without stopping.
If $b < c$, population declines to zero.
If $b = c$, constant population.
6: Compare with reality
criticize model
Given various data on populations — sometimes reasonable fit.
Improvement of model needed.
7: Write a report

2.2 Modelling skills exercises

The seven-box diagram is a way of breaking down modelling into a number of activities. As you know from *Unit M*, various skills underlie these activities. One of the activities is the formulation of a mathematical model. The exercises below illustrate some of the skills required to do this, and serve as revision of the work you did in *Unit M*. Do not spend more than about ten minutes on any exercise.

The solutions to these exercises given at the end of the booklet are not intended to be definitive, and you may disagree with some of them. It is characteristic of mathematical modelling that there may be several alternative models to use in a given problem; your approach to a problem may be different from that of somebody else.

What you should aim for is a model that enables you to describe and make predictions about those features of the problem you choose to emphasize. So do not worry if your answers to these modelling skills exercises differ from those given, provided that you have not made an obvious mistake.

Exercise 1

Generating variables

The headteacher of a new school is trying to decide how long the lessons should be. Write down all the features of the situation that may need to be considered in making a decision.

Exercise 2

Selecting variables

You have read an eyewitness report of a collision between two cars, and you are trying to estimate from this report the likely cost of repairing damage to the vehicles. The report contained the features listed below:

(i) the age of each driver;

(ii) the angle of impact between the cars;

(iii) the alcohol content of each driver's blood;

(iv) the current value of each car;

(v) the make and model of each car;

(vi) the reaction time of each driver;

(vii) the mass of each car;

(viii) the speed of each car at the time of impact;

(ix) the weather conditions at the accident scene.

Give each feature a rating to indicate how important that feature is to the making of a good estimate of the cost of repairing the damage to the cars, using the following rating scale.

Rating	*Importance of feature*
1	Key factor: most important, must be considered.
2	Secondary factor: may need to be considered.
3	Very minor factor: unlikely to be considered, except in a most detailed analysis of the problem.
4	Not a factor: should not be considered.

Exercise 3

Formulating questions

A traffic engineer is analysing the problem of phasing a new set of traffic lights which have been installed at a normal crossroads (so that there are no right- or left-turn 'arrows').

(i) What information must the engineer specify in order to set the lights in operation?

(ii) What features of the traffic conditions will the engineer take into account to determine this information?

Exercise 4

Generating relations

A car sales assistant is paid each week a set amount of commission for each car that is sold during the week *or* a basic weekly salary, whichever is the greater. Sketch a graph showing how the sales assistant's weekly earnings vary with the number of cars sold.

Exercise 5

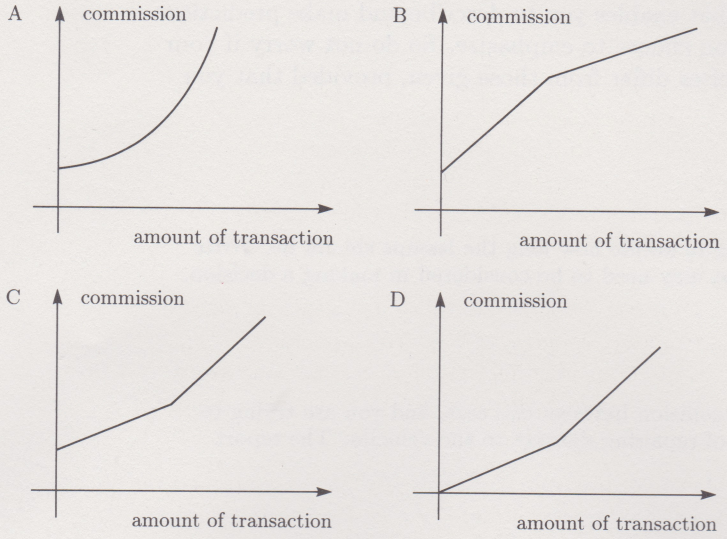
Selecting relations

An agent receives commission for each transaction he or she makes as follows:

(i) for transactions less than £30 000, £750 plus $2\frac{1}{2}\%$ of the transaction;

(ii) for transactions more than £30 000, 5% of the transaction.

Which of the following graphs shows this situation most realistically?



[*Solutions on page 29*]

2.3 Getting started

A mathematical model is a set of relations between variables representing the main features of the situation being modelled. Before one can begin to formulate a model it is therefore necessary to select some appropriate variables. This may be difficult if one is dealing with a situation which is at all complicated. In this subsection we shall describe a technique for getting started on a modelling problem which many people find helpful. The technique uses 'brainstorming' to produce a list of features of the situation; using this feature list, one can usually see how to select some appropriate variables and begin to get an idea of how they might be related.

This approach to getting started builds on two of the modelling skills considered in the last subsection, namely

- identifying the important features of a problem, and
- deciding which of these features should be included in the first very simple model.

Exercise 1 illustrated the first of these skills. The features that the headteacher may need to consider in making a decision about the length of lessons seem quite obvious. You may have added to or omitted some of the features listed in the solution. However, the list given in the solution is only a beginning. Some features will be more important than others, so that, as a first step, a simple mathematical model would include only what you consider to be the most important features. The second skill, of rating the relative importance of different features, was illustrated in Exercise 2. Once you have generated a list of features of the problem, you have then to prune the list to leave only the important ones. For example, the 'reaction time of each driver' surely has nothing directly to do with the cost of repairs to each vehicle, and so may safely be omitted from the feature list.

Exercises 1 and 2 have given clues as to how to get started on more complicated problems. They suggest the following technique.

1. *List* as many features as you can which have something to do with the problem. This is the 'brainstorming' part of the activity: you should not pause to analyse the features as you write them down, but simply aim to generate a stream of ideas. As a result, your list may include superficial and even irrelevant ideas; but write them down anyway, because sometimes something which is superficial or irrelevant may lead you to think of another more important feature.
2. *Refine* this list by omitting everything except the features which are relevant and 'sensible'. Then try to link together those of the remaining features which influence each other. In contrast to the first step, this is a reflective and analytical process.

3. *Prune* the refined feature list, so as to leave only a very few features that you regard as absolutely essential. Remember that your aim is to identify just two or three key variables which will appear in your model. It will probably be necessary to be quite ruthless with this pruning. In deciding which features to keep and which to ignore to begin with, you will have to make simplifying assumptions: for example, you may decide that one feature rather than another has a far more significant effect on the process under consideration. Remember also that if you are going to represent a feature in your model by a variable, you must be able to quantify it. While you are carrying out this pruning you should bear it in mind that modelling proceeds by stages: your first model should be as simple as possible, but it will form the basis for further models which will be better and more complex. The assumptions you have made in pruning your feature list, and the features you have decided to omit at this stage, will be useful later when you come to criticize and improve your initial model – so keep a note of them.

We shall next present a couple of problems to illustrate how this technique is applied in practice. The first will be a demonstration of the technique, the second will give you an opportunity to practise it for yourself. There is, of course, no correct answer in an exercise like this, and you will have to bear this in mind as you work; your aim should be to get an idea of the kind of thing to include in a feature list and what to retain as you refine and prune it.

Example 1

A deer park is about to be opened in the grounds of a stately home. It will be stocked with a small herd of deer. As the deer breed, the size of the herd will increase, and if no action is taken to control the number of deer there is a danger that the park will become overgrazed. How many deer should be culled (i.e. killed) each year to prevent overgrazing?

Solution

Here is an initial feature list, compiled simply by writing down features of the problem as they came to mind.

- initial size of herd
- size of the park
- rate at which the deer eat grass
- number of males and females
- age distribution of the deer
- number of years the deer remain fertile
- death by old age
- types of predators
- birth rate
- population of predators
- diseases
- proneness of deer to fight in an enclosed environment
- rate at which the grass grows
- effects of being in a park instead of in the wild
- 'family' size
- role of unattached males
- sex of deer to be killed
- when to kill the deer
- do deer pine if their mates or offspring are killed?
- is there an alternative food to grass?

As you can see, this is quite a mixed bag, ranging from fundamental features, like the birth rate, to merely fanciful ones, like the reaction of the deer to the deaths of mates or offspring. But most of these features can be assigned to one of just two convenient groups: features which affect the growth of the deer population, and features which affect the availability of grass.

The following features affect the growth of the deer population.

- initial size of herd
- number of males and females
- age distribution of the deer
- number of years the deer remain fertile
- death by old age
- types of predators
- birth rate
- population of predators
- diseases
- sex of deer to be killed
- when to kill the deer

These are the features that are concerned with the availability of grass.

- size of the park
- rate at which the deer eat grass
- rate at which the grass grows

The remaining features from the original list are as follows.

- proneness of deer to fight in an enclosed environment
- effects of being in a park instead of in the wild
- 'family' size
- role of unattached males
- do deer pine if their mates or offspring are killed?
- is there an alternative food to grass?

By grouping the features in this way we begin to get an idea of how to approach the problem. We have to find what size of herd the park can sustain: this we should be able to calculate using the features that affect the availability of grass. We can then model the growth in the size of the herd due to the annual births and natural deaths, and determine how many deer must be culled to keep the herd to the size that the park can sustain. If we take this approach then the miscellaneous features in the final list above are all irrelevant, and may be ignored. The lists of features affecting the growth of the deer population, and of features concerned with the availability of grass, together constitute our refined and grouped feature list. However, if we try to take into account all the features that we have identified as affecting the growth of the deer population, we shall have a very complicated model to deal with: so there is still pruning to be done. Remember that it is important to begin with as simple a model as possible. For a pruned feature list we might therefore take the following.

Features affecting the growth of the deer population

- initial size of herd
- number of males and females
- death by old age
- birth rate

Features that are concerned with the availability of grass

- size of the park
- rate at which the deer eat grass
- rate at which the grass grows □

To sum up: the final feature list is obtained in three steps, as follows.

1. Make an *initial* feature list by noting down as many things as come to mind. Do not criticize any feature at this stage.
2. Obtain a *refined* list by rejecting any irrelevant features that occurred in the initial list. In this list, related features may be grouped together.
3. Derive a *pruned* list, consisting of the essential features that permit the construction of a simple mathematical model of the problem.

Here is a problem for you to try.

Exercise 6

This problem is about the athletics event of shot-putting. There are two important rules for the shot-putter: the shot must be 'pushed' from the shoulder, not thrown; and the shot-putter must remain within a circle of diameter 7 feet. The aim of the athlete is to make the shot travel as far as possible before it hits the ground. How can the shot-putter achieve the maximum distance when putting the shot?

(i) Draw up an initial feature list for this problem.

(ii) By first grouping the features in your initial list into those which affect the motion of the shot through the air, and those which affect the shot-putter's action, obtain a refined and grouped list.

(iii) Prune your refined list so that the final list contains not more than five features under either heading (i.e. not more than five features which affect the motion of the shot through the air, and not more than five which affect the shot-putter's action).

[*Solution on page 29*]

2.4 Choosing a problem to model

In the previous subsection we investigated two problems and found three steps that helped sort out the essential features of each problem. The three steps are as follows.

Step 1: a feature list
Step 2: a refined and grouped list
Step 3: a pruned list

To help you choose a problem for your own modelling work for TMA 07 from the various specifications available, we suggest that you apply these three steps to each problem specification. When you have done this, fill in the problem choice table given on page 13. In the first column of this table, write a single sentence stating the problem to be solved. Put these problems in order of interest by completing the second column. Now transfer your pruned feature list into the third column. You should also make a note of the simplifying assumptions that you have made in rejecting some features from the initial feature list.

The problem specifications are given in the current Assignment Booklet.

Choosing a problem or research topic is always difficult, and none of the problems specified in the Assignment Booklet may seem ideal. We suggest that you choose the most interesting problem for which you find yourself best able to list simplifying assumptions.

To show how the problem choice table is intended to be used, we have given an example based on the three problems below.

Note that the three problems stated below are *NOT* available as problem specifications for you to work on for TMA07. Your problem specifications are given in the current Assignment Booklet.

Problem specification A

Investigate the comparative economics of

(i) buying and installing a time-switch for your domestic hot water,

(ii) buying additional insulation for the hot-water tank,

(iii) just leaving the hot water on all the time.

For the purpose of this project: (a) assume that you do not already have a time-switch; (b) otherwise use your own system; (c) solid fuel systems with 'energy to waste' do *not* qualify.

Problem specification B

Discuss whether taking an OU degree is a financially worthwhile activity in the long term. Does the award of the degree enhance your salary and prospects sufficiently to make up for the total cost of taking the degree?

How does the economics of an OU degree compared with that of a conventional university degree taken

(i) as a mature student,

(ii) when you are 18?

Problem to be solved (stated as a single-sentence question)	Order of interest (1 = most)	Important features and simplifying assumptions
(A) To compare methods of heating hot water and look at the costs involved	2	cost of heating water, how much water is used, local weather, installation and maintenance costs, heat loss, thickness of insulation Assume constant fuel costs, constant maintenance costs, average temperatures, no radiation
(B) Is it cheaper to do an O.U. degree or from another institution?	3	loss of earnings, cost of course and extras, financial reward at the end, mature student at conventional university. Comment Not keen on this one. can't write down much about it
(C) To develop a model of safe-driving distance	1	car speed, car weight, road surface, weather conditions, traffic density driver health and reaction time tyre type, vehicle spacing Assume straight road, fit driver vehicle functioning well

Example for Problem specification C

Problem specification C

Develop models for the concept of a 'safe driving distance' under a variety of assumptions and traffic conditions. Discuss the giving of official guidance to motorists. Compare your results with what actually happens in practice.

An example of a completed problem choice table is given on page 12. It was filled in by a student who was trying to decide which of the above problems to attempt. It took the student about 1 hour to fill in; the problem chosen was the one with specification **C**.

Problem choice table

Complete this table for the problem specifications given in the Assignment Booklet.

Problem to be solved (stated as a single-sentence question)	Order of interest (1 = most)	Important features and simplifying assumptions

3 At Summer School

When you arrive at Summer School you will be asked which of the modelling problems you wish to work on for TMA 07. The exercises in the last subsection should have prepared you for this, and also given you some ideas as to how to develop a model for the problem you have chosen. While you are at Summer School you will be working in a group under the guidance of a tutor, so you will be able to share your ideas with other students. The first activity at Summer School will therefore be to produce a common feature list by pooling the lists of all the students in your group. This will be the basis for your subsequent work at Summer School. The groups will be arranged so that all students in a group are working on the same problem, of course. But in addition, we shall try to ensure as far as possible that the other students with whom you will be working most closely have identified the same kind of approach to the problem as you have, and the work you have done before Summer School in deriving a pruned feature list for the problem will help here as well.

By the end of your Summer School week you should at the very least have developed a first model to the point where you can see how to answer the problem posed; the Summer School programme has been organized to ensure this. Since your work will be supervised by a tutor, there is no need to go into detail here about how to model; besides, the processes involved have been discussed at some length in *Unit M*. But as a reminder, and to help you get your mind focussed on modelling again, we reprint below the description of the main steps one goes through in developing a mathematical model from the summary of Section 4 of *Unit M*. If you have Summer School excusal, you should find this summary particularly helpful, and you are strongly advised to reread the whole of Section 4 of *Unit M* before attempting your chosen modelling problem for TMA 07 by yourself.

1. Specify the real problem
   - Decide what process you will have to model in order to answer the question posed.
2. Set up a model
   - Pick out the key variables that will have to be used in the model.
   - Try to formulate the underlying modelling problem as 'how does one of my variables depend on the other(s)?'.
   - Use your common sense and experience to say what you can about the qualitative aspects of the relationship between your key variables.
   - Formulate a relationship between your variables, by using relevant physical laws, principles like the input-output principle, your own mental picture of the processes involved, and any similar models you have met before, remembering always to simplify the problem as much as you can.
   - State what simplifying assumptions you have made in formulating your model.
3. Formulate the mathematical problem
   - Re-interpret your initial problem in terms of your model; convert the modelling problem into a mathematical one ...
4. Solve the mathematical problem
   - ... and solve it.
5. Interpret the solution
   - Decide what data you will need in order to specify the values of any parameters, and collect them.
   - Make any qualitative checks you can think of, to confirm that you are on the right track.
   - Interpret your mathematical solution so as to give a solution to the original problem in appropriate terms.

6. Compare with reality
   - Decide what data you will need to validate your model, and collect them.
   - Think of ways of improving your model, by relaxing one or more of your simplifying assumptions.

It is worth emphasizing again the importance of making your first model simple. In particular, if at step 3 you find you have produced a mathematical problem which you do not know how to solve, go back to step 2 and simplify your model further.

You are strongly advised to keep comprehensive and accurate notes of the work you do at Summer School. In particular, make sure you have a record of your list of variables, your assumptions, the equation or other relationship which constitutes your model, and your solution to the problem.

4 After Summer School

There are two weeks set aside in the course calendar after Summer School for you to complete your work on your model and prepare your report for TMA 07. You should expect to spend the first of these weeks on further work on the model and the second on writing your report.

4.1 Finishing off your modelling work

You may still have to put some finishing touches to the main model that you developed at Summer School. It should not take you long to complete the work as far as obtaining a solution to the original problem, even if you did not quite reach this stage while at Summer School. However, this is not the end of the exercise. It is important to evaluate your results, and the model that produced them. You will want to assess how accurate and reliable your results are, and for this purpose you may need to collect more data about the situation you are modelling, either by consulting books or similar sources, or possibly by carrying out a simple experiment or engaging in some other form of observation. Please do not ask other people or institutions to provide you with data. Remember, in particular, that many other students are engaged in the same activity as you are; imagine the effect if they all write simultaneously to the same body asking for information.

You will have been very fortunate – or very skilful – indeed if your results prove to be entirely satisfactory when you compare them with reality, since your first model should have been a simple one. You will therefore want to revise and improve your model, which will involve another circuit of the loop in the modelling diagram. One good method of finding a way of revising your model is to review the simplifying assumptions that you made when formulating it. Choose one of those assumptions, and try the effect on your model of deleting it or relaxing it. Alternatively, you could try to incorporate one of the features that you eliminated when pruning your feature list.

When you have decided how to revise your model, you will have to carry out the necessary modifications, solve the new mathematical problem, and re-evaluate your results to see whether your revisions did lead to improvements.

4.2 Writing your report

It is almost too obvious to need saying, that your purpose when you come to write your report of your modelling work is to explain to your tutor what results you obtained and how you obtained them, and to do this clearly and fully so that he or she will be left in no doubt about what you have achieved. After all, your tutor cannot award marks for work he or she does not understand, nor can he or she be expected to guess what you did if you have not written it down. It is your responsibility to explain your work satisfactorily. TMA 07 is really no different from any other TMA in this respect. The difference is that solutions to the questions in other TMAs (except TMA 04) mostly follow a pretty straightforward pattern which is dictated by the question. In the case of TMA 07, however, you apparently have a great deal of freedom in the way you organize your report and in your choice of words and symbols. The purpose of this subsection is to give you some advice about how to write your report.

Let us start by reassuring you that we are not setting exacting standards of literary style. We are not even going to worry unduly about occasional spelling mistakes. We are concerned here with the aspects of writing which are closest to mathematics: being systematic; laying out your work in a logical fashion; making sure that each step of an argument or calculation follows from the previous one; avoiding any obscurity or ambiguity in the use of symbols and technical terms; using appropriate diagrams and tables. However, no piece of writing is going to make much sense if it contains lots of meaningless or ambiguous sentences, so you must make sure that what you write is clearly expressed.

There will undoubtedly have been occasions on which you have been annoyed or inconvenienced by some piece of technical writing which was badly expressed: you may have had to puzzle for hours over the instruction manual for a calculator, or the documentation for some computer software, or even the solution to one of the exercises in this or another course. There will also have been occasions on which you have quickly achieved understanding of something through reading a clear and well-written explanation of it. Try to recollect a specific example of a short piece of bad writing, and one of good writing. (It may be easier to think of an example of bad writing than one of good: one of the characteristics of good technical writing is that it does not draw attention to itself.) Now imagine that you are an MST204 tutor having to mark a piece of work written in a similar style to your bad example, and compare your attitude to it with your attitude to a TMA as well written as your good example. It surely does not require much thought to decide which of the two pieces of work will be easier and more pleasant to mark.

It is a useful exercise to try to list the things that make it difficult to follow a badly-written technical document. Here are some features of bad writing.

- The problem which the document is intended to deal with is not sufficiently clearly described, so that the reader is left wondering what it is all about.
- There is no clear and logical line of development – there are many digressions, and the points made do not appear to relate to each other.
- There are too few 'signposts' explaining where the text is leading: the reader is unable to see the wood for the trees. This can be a fault of an otherwise well-structured piece of writing.
- Assumptions are made implicitly, without being mentioned in the text. One common symptom of this fault is that terms mysteriously disappear from, or appear in, equations without explanation.
- Technical terms or symbols are used without explanation, so that the reader has to guess what they mean. Just as bad is for a technical term or symbol to be defined ambiguously, or used with different meanings in different parts of the document.
- There is inadequate explanation of how one step in an argument follows from the previous one, or of how a calculation is carried out, so that the reader is left uncertain as to how the result is derived.
- Opportunities are missed to simplify complicated explanations by using diagrams.
- Data which play an important role in understanding the results are difficult to find because they are scattered through the text rather than being presented in a convenient table.

There are probably other faults which you have noticed in documents you have read, which you could add to this list; you should note them down, and try to make sure that your writing does not contain such faults. However, the list above does include many of the worst faults of bad writing, so we shall concentrate here on suggesting how these may be avoided.

One of the most important things to get right when writing a technical report is to ensure that it has a clear and comprehensible structure. The first three of the faults listed above are concerned with lack of an adequate structure. We are now going to suggest a structure for your report for TMA 07. If you follow this suggestion, at least in broad terms, then you should be in no danger of making the errors described in the first few items of the list above. We should also point out that the mark scheme for TMA 07 is based on the structure we are about to describe – more on this in the next subsection.

Broadly speaking, the main body of your report should follow the stages set out in the seven-box modelling diagram. First, there should be an introduction, whose purpose is to ensure that the reader has a clear understanding of the problem being tackled, and to give him or her a signpost to the remainder of the report. This will be followed by an explanation of how the model is formulated; an account of the mathematical procedures you have used to solve the problem; and the interpretation of your solution.

You should then make a critical evaluation of your first model and its results, and describe your attempts to make improvements. You should conclude with any suggestions you can make for further work that could be done on the problem.

Here is a more detailed description of the suggested structure for your report. The report should be organized under seven subheadings, as shown below; for each subheading we have given the points that should be covered in the corresponding section of the report.

Introduction

1. A statement of the problem dealt with in the report, in your own words.
2. Your assessment of the significance of the problem.
3. A brief outline of your approach to the solution of the problem.
4. A summary of your conclusions, or a representative sample of the results you have obtained as your solution to the problem.

Formulation of the mathematical model

1. A description of the main features of the process being modelled.
2. A list of the simplifying assumptions you have made.
3. A careful definition of each of the variables used in the model.
4. An outline in words of how your model is derived, and how it follows from the assumptions.
5. Your formulation of the equations or other mathematical relations which constitute the model.

The mathematical problem and its solution

1. A statement of the mathematical problem to be solved, and an explanation of how it is related to the problem posed in the Introduction.
2. The solution of the mathematical problem.

Interpretation of the solution

1. A qualitative interpretation of your solution to the mathematical problem, including any qualitative checks that can be made.
2. The calculation of the values of any parameters in the model, including a statement of any data required and a description of your sources for them.
3. The quantitative solution to the problem posed in the Introduction.

Evaluation of the model

1. A description of any tests of the results that have been carried out, including an explanation of how the necessary data were obtained.
2. An assessment of the reliability of the quantitative solution to the problem.
3. The identification of any shortcomings of the model.

Revisions to the model

1. A revised model, based on the criticisms of the initial model.
2. The solution of the revised mathematical problem.
3. An evaluation of the results of the revised model.

Conclusions

Let us now go back over the structure outlined above, and describe in a little more detail what is required under each subheading.

Introduction

1. *A statement of the problem dealt with in the report, in your own words.*
   You should not merely repeat the problem specification as it is given in the Assignment Booklet, but reformulate it so as to focus on the aspects of the problem you have chosen to concentrate on.
2. *Your assessment of the significance of the problem.*
   Say something about the background to the problem, and how general the problem is or could be. If you think it is important or useful to solve this problem, say why; if not, say why not.
3. *A brief outline of your approach to the solution of the problem.*
   Say, in a couple of sentences at most, what kind of model you have developed, what principles you have applied to derive it, and so on.
4. *A summary of your conclusions, or a representative sample of the results you have obtained as your solution to the problem.*
   Give your reader an idea of the results you have obtained, including a foretaste of the actual numerical results if appropriate.

Formulation of the mathematical model

1. *A description of the main features of the process being modelled.*
   Do *not* reproduce your initial feature list (if you made one): that was merely a means to an end. A final pruned and grouped list would be suitable, however. You may wish to mention any features that you considered seriously but did not include in your main model in the end; you can explain why you excluded them in your simplifying assumptions.
2. *A list of the simplifying assumptions you have made.*
   It should be possible for you to justify each major step in the formulation of your model by appealing to one of your assumptions. On the other hand, an assumption which is not necessary for the derivation of the model is superfluous and should be omitted.
3. *A careful definition of each of the variables used in the model.*
   You may decide to list all the variables (and parameters) together, perhaps in a table; or you may think it is better just to give the main ones here and introduce others where they are needed in the development of the model. In either case, make quite sure that each variable is clearly and unambiguously defined, and that its units are given if appropriate.
4. *An outline in words of how your model is derived, and how it follows from the assumptions.*
   Explain to your reader, in words, how you are going to obtain your model. This description should make the derivation of the model, when it comes, easy to follow. You will probably want to refer frequently here to the list of simplifying assumptions.
5. *Your formulation of the equations or other mathematical relations which constitute the model.*
   This is probably the most difficult part of the report to do well, so make sure you re-read what you have written with a critical eye to ensure that it is clear. If you have found a captive audience to try out your report on, this is where their help will be most useful: ask your audience whether your description of the formulation of your model is easy to follow.

The mathematical problem and its solution

1. *A statement of the mathematical problem to be solved, and an explanation of how it is related to the problem posed in the Introduction.*
   Tell your reader, in words, what mathematics you are going to do, and how it will help to solve the problem.
2. *The solution of the mathematical problem.*
   This should be easy, because it is pretty much like writing out your solution to an ordinary TMA question. You may assume any standard mathematical results you need, such as those in the *Handbook*.

Interpretation of the solution

1. *A qualitative interpretation of your solution to the mathematical problem, including any qualitative checks that can be made.*
   If, for example, your solution is a function, you may want to sketch its graph, or say whether it is increasing or decreasing. You could also consider how your solution compares with what common sense suggests, or show that it reproduces known results in simpler situations or limiting cases.
2. *The calculation of the values of any parameters in the model, including a statement of any data required and a description of your sources for them.*
   You should give references to any sources of data you have used, and descriptions of any measurements you have carried out yourself.
3. *The quantitative solution to the problem posed in the Introduction.*
   Make sure that you choose an appropriate level of accuracy for your numerical answers. It would be worthwhile thinking of a way of presenting your results which is appropriate to the problem in hand. If you have a lot of results to record, you should consider how to make it easy for the reader to comprehend them, by putting them in a table for example.

Evaluation of the model

1. *A description of any tests of the results that have been carried out, including an explanation of how the necessary data were obtained.*
   You should try to test some predictions of your model against real data, to get some idea of how accurate and reliable your model is. Again, you should give references to books or other sources of data, and if you have carried out any simple experiment you should describe your equipment and procedure. However, it sometimes happens that testing a model is impossible, either for theoretical or for practical reasons; if you think that this is true of your model, you should explain why.
2. *An assessment of the reliability of the quantitative solution to the problem.*
   You might consider the qualitative effects of some of the simplifying assumptions you have made: is the simplification likely to lead to an answer that is too large or too small? For a really impressive report you could work out how sensitive your results are to the accuracy of the values of the parameters.
3. *The identification of any shortcomings of the model.*
   Criticism of one's work is a healthy practice, if carried out in moderation, and this is what you are being invited to do here. You may find it helpful to consider any tests of your model that you have carried out, in relation to the simplifications you made in formulating the model. Don't forget to mention the positive points about the model as well as the negative ones.

Revisions to the model

1. *A revised model, based on the criticisms of the initial model.*
   Your best policy for revising and improving your model is to relax one or more of the simplifying assumptions you have made, and in writing this section of the report you should make it clear which assumptions have been changed. You need not go through all the details of the formulation of the model again; just change what has to be changed as a consequence of the new assumptions.
2. *The solution of the revised mathematical problem.*
   Again, you can probably modify the work that you have already done in solving your first model.
3. *An evaluation of the results of the revised model.*
   The important point here is to discuss the extent to which your modifications have produced an improvement in the results.

Conclusions

This is your opportunity to sum up the work that you have done, to say whether or not you have obtained a satisfactory solution to the problem you set out to solve, and if not to make suggestions for further improvements.

We shall end this subsection with a few general suggestions as to how to set about writing your report.

Your report will actually be read (and marked) by your tutor, of course; but we suggest that in writing it you imagine that it is to be read by another student. This should allow you to make sensible judgements about the amount of mathematical detail to include.

Although we suggest that your report should have the structure described above, this does not mean that it must be written in the order in which it will be finally presented and read. Most people find it quite difficult to write anything longer than a postcard by starting at the beginning and keeping steadily on until the end is reached. It is usually necessary to make at least one draft, and to work backwards and forwards through the draft as one's ideas develop. The first section of the final version of the report is supposed to be an introduction to the rest of it; so it is actually sensible to write this section last, when you will know exactly what there is to introduce. The best place to start is probably in the middle, with the formulation of the equations of your model. As you work on this part you will introduce the symbols for your variables: note them, and their definitions, separately and you will have the nucleus of your list of variables. As you explain how your model is derived you will call on various simplifying assumptions: note them separately for inclusion near the beginning of the formulation section.

If you proceed in this way, building up your report on several fronts simultaneously, you will find that the resulting account of your model is more comprehensive and better structured than if you just start with the introduction and carry on to the conclusion. If you are fortunate enough to have access to a word processor you will find this approach especially easy; but it is not difficult with pen and paper.

Take every opportunity to include figures and illustrations in your report. You can give sketches of graphs, drawings of experimental equipment, and diagrams of processes and relationships. Explanations of complicated derivations are often easier to understand if they are accompanied by appropriate diagrams. Data, numerical results and even variables are usually best displayed in tables.

If you are aware of the common faults of technical writing, if you use the structure we have recommended, and if you follow the suggestions above, you should produce an excellent report.

4.3 Mark scheme for TMA 07

> Your report should be between 1500 and 2500 words in length. On no account should it be longer than 3000 words in total; *if it is, your tutor will mark and comment on only the first 3000 words.*

The mark scheme for TMA 07 is based on the structure for the report which was described in the previous subsection. You are not required to follow that structure to the letter; how you write your report will depend to some extent on the problem you have chosen and the model you have developed. However, the points raised in the discussion in the previous subsection should all appear somewhere in your report; and clearly it will be in your interests to follow our suggested outline as closely as possible because it will make it easier for your tutor to mark your work.

In fact, the structure of the sample report in the next subsection differs slightly from the recommended one: it turned out to be more natural to deal with testing the results, and the calculation of parameters, together.

The marks for your report will be apportioned to the various section headings as follows.

Section	Marks
Introduction and Conclusions	8
Formulation of the mathematical model	20
The mathematical problem and its solution	16
Interpretation of the solution	20
Evaluation of the model	12
Revisions to the model	16
Total	92

There remain 8 marks which are to be awarded for originality and good presentation, at your tutor's discretion. The intention is to give your tutor the opportunity to reward any especially noteworthy aspect of your work which is not covered by the rest of the mark scheme. Discretionary marks will be awarded for things like: a particularly ingenious model; a well-designed experiment; clever use of a computer to solve the problem; good use of diagrams or tables; an especially well-presented report. The discretionary marks will have to be earned, so do not expect to be awarded any unless you have made a special effort on some part of the modelling exercise.

4.4 A sample report

To give you some idea of what a report written with our suggested structure looks like, we have edited the account of the drug therapy model from *Unit M* Subsection 2.1 to bring it into line with that format. We have extended the model by adding some of the results of the exercises which were set on it; this has allowed us to include an evaluation of the model and revisions to it. The report is not to be regarded as perfect; its function is to give you guidance on the layout and length of your own report (this one is about 2500 words long). The comments in the margin should help you to compare the report with the outline given in Subsection 4.2 above; you are not expected to include similar comments in your own report.

The data for this model were obtained from a national institution, so we appear to have broken the rule set out in Subsection 4.1. However, this is a somewhat special case, since the report originated as the programme notes for a television programme. We remind you that the following rule applies to your own modelling work.

A note on data collection for your problem

You should use only data which is publicly available (e.g. in libraries) or which you collect yourself. Please do not write to national bodies, research institutes, etc.

The frequency of administration of doses of aminophylline in the treatment of asthma

Introduction

Statement of the problem

Patients with asthma suffer from constriction of the airways in the lungs and, consequently, difficulty in breathing out. This ailment can be alleviated by introducing the drug theophylline into the bloodstream. This is done by injecting another drug, aminophylline, which the body quickly converts to theophylline. Once present in the blood, the drug is steadily excreted from the body via the kidneys, and unless there is replenishment, the quantity of drug in the blood will fall.

It is known from experiment that theophylline has hardly any therapeutic effect if its concentration in the bloodstream is below 5 mg/l (milligrams per litre), and that concentrations of above 20 mg/l are likely to have toxic effects. The problem is to administer the drug in such a way that the concentration remains within the therapeutic range between 5 mg/l and 20 mg/l.

Its significance

It is clearly of great importance in the treatment of any illness that the drugs which are prescribed should be given in the correct doses at the correct intervals. The specific problem considered here is one of a general class concerned with the use of drugs in the treatment of a wide variety of diseases.

Brief outline of the approach used

The approach adopted is to use the input-output principle to derive a differential equation for the quantity of drug in the bloodstream, from which its variation with time may be derived.

Conclusions and results

The results obtained may be summarized in the form of possible instructions to the nursing staff, as follows.

1. Weigh the patient on admission.
2. Give an initial dose of 60 mg of aminophylline for every 10 kg of the patient's weight.
3. Give a booster dose of half the initial dose every four hours.

A better method of administering the drug, using continuous infusion, is also analysed; the same initial dose is given, but subsequently the drug is infused at about 2 mg/hour.

Formulation of the model

Main features to be modelled

The main feature to be modelled is the process by which the drug is eliminated from the body. This takes place in the kidneys, which act as filters removing impurities, including the drug, from the bloodstream.

Simplifying assumptions

The model is based on the following assumptions.

(a) The conversion from aminophylline to theophylline takes place at the moment of injection. This is equivalent to assuming that it is theophylline itself which is injected.

(b) The theophylline is distributed uniformly throughout the bloodstream upon injection and at all subsequent times. Hence the drug's concentration at a given time is the same at all points of the bloodstream.

(c) The quantity of theophylline in the body is proportional to its concentration in the blood. The constant of proportionality is called the *apparent volume of distribution.*

(d) No theophylline enters the body between injections.

(e) Every molecule of the drug has the same chance of being filtered out during its passage through the kidneys. The rate of flow of blood through the kidneys is assumed to be constant.

Definition of the variables

The main variables used are

- t the time in hours after the administration of the drug
- c the drug concentration in the blood in mg/l.

Other subsidiary variables and parameters will be defined as they are needed.

Outline of the model

The main aim of the model is to find how the drug concentration c varies with time t. (The concentration is a function of time alone, because of assumption (b).) To do this, assumptions (d) and (e) are used in concert with the input-output principle, which is

$$\boxed{\text{accumulation}} = \boxed{\text{input}} \text{ minus } \boxed{\text{output}}.$$

Let q be the quantity of theophylline present, in mg. Assumption (c) can be expressed as

$$c(t) = \frac{q(t)}{V},$$

where V is the apparent volume of distribution, in litres.

Formulation of the equations

The input-output principle is applied to the quantity of theophylline in the body for an interval between times t and $t + \delta t$ during which no injection is administered. The accumulation of the drug over this interval is equal to the difference between the quantities present at the end and at the start of the interval, that is,

$$\text{accumulation} = q(t + \delta t) - q(t).$$

By assumption (d) there is no replenishment of the drug during the interval, and so

$$\text{input} = 0.$$

Let the rate of flow of blood through the kidneys be ρ litres per hour. Then in a time interval of length δt hours, the volume of blood flowing through the kidneys is $\rho\delta t$ litres, and the mass of drug passing into the kidneys is thus approximately $c(t)\rho\delta t$ mg. (Note that this deduction holds only for short periods of time, so that the variation of $c(t)$ with time t can be ignored.)

The number of drug molecules in this quantity of drug is proportional to the mass of drug, and since each molecule has the same chance of being filtered out (assumption (e)), a fixed proportion of this mass of drug is excreted. Thus

$$q(t + \delta t) - q(t) \simeq -kc(t)\rho\delta t,$$

where k is the constant of proportionality. But $c(t) = q(t)/V$, so

$$q(t + \delta t) - q(t) \simeq -\lambda q(t)\delta t,$$

with $\lambda = -k\rho/V$. Dividing through by δt and taking the limit as $\delta t \to 0$ gives

$$\frac{dq}{dt} = -\lambda q.$$

This differential equation for $q(t)$ can be supplemented by an initial condition. If a dose D mg is injected at time $t = 0$ into a patient who was previously free of the drug then, according to assumption (a), we obtain

$$q(0) = D.$$

The mathematical problem and its solution

Statement of the mathematical problem

Given the differential equation

$$\frac{dq}{dt} = -\lambda q \qquad (t > 0)$$

and the initial condition

$$q(0) = D,$$

we must find an expression for $q(t)$, then use the relation

$$c(t) = \frac{q(t)}{V}$$

to deduce an expression for $c(t)$. Then knowing how the drug concentration depends on the time, we shall be able to predict how long it will take to drop below the limit of effectiveness.

Solution of the mathematical problem

The general solution of the differential equation is

$$q = Ae^{-\lambda t},$$

where A is an arbitrary constant. Putting $t = 0$ into the general solution and using the initial condition $q(0) = D$ gives

$$D = A,$$

so that the required particular solution is

$$q = De^{-\lambda t},$$

and therefore

$$c = \frac{D}{V}e^{-\lambda t}.$$

Interpretation of the solution

Qualitative interpretation

A qualitative prediction of the exponential decay solution is that between injections the concentration will fall with time but more and more slowly.

The form of the solution is itself a quantitative prediction, but it can be put into a form more amenable to comparison with reality. Taking the logarithms of both sides of the equation for c gives

$$\log_e c = \log_e \left(\frac{D}{V}\right) - \lambda t.$$

This predicts that a graph of $\log_e c$ against t should be a straight line.

Testing the results

Experimental measurements can be obtained by injecting a known dose of aminophylline into a patient, allowing time for it to diffuse throughout the bloodstream and subsequently finding the concentration of theophylline in blood samples taken at regular intervals. The diagram below shows a graph of concentration against time which could be obtained from such an experiment.

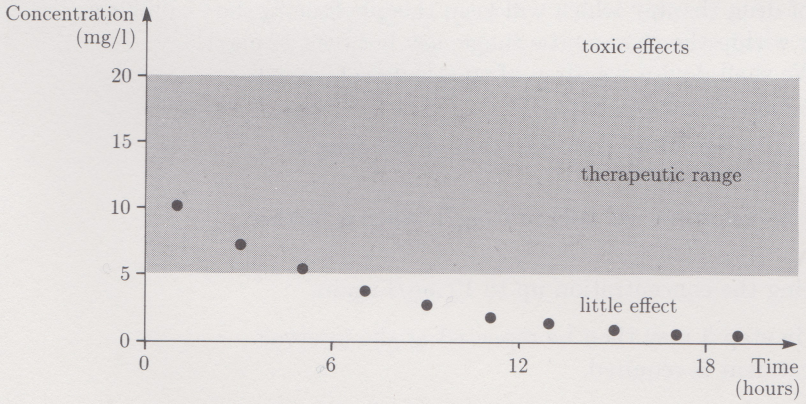


Figure 1

The qualitative prediction is satisfied by the experimental results shown on the graph in Figure 1. The quantitative prediction is also satisfied, for when the experimental data are plotted as suggested, the result is a straight line, as shown in the following figure.

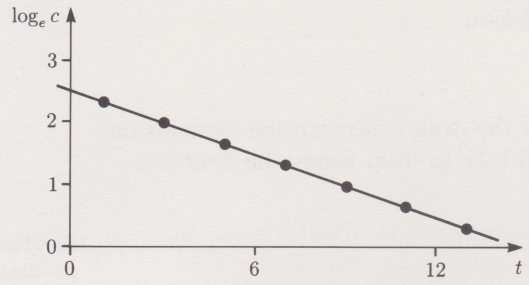


Figure 2

Calculation of the values of parameters

To two significant figures, the slope of this line is -0.17 and the intercept on the vertical axis is 2.5. Thus

$$\lambda = 0.17,$$

and

$$\log_e\left(\frac{D}{V}\right) = 2.5$$

or

$$\frac{D}{V} = e^{2.5} \simeq 12.$$

Thus our model says that the concentration c is given by

$$c = 12e^{-0.17t},$$

so that it decreases steadily.

Experiments show that there is little variation in the value of λ from patient to patient. We shall therefore continue to use the value 0.17 for λ.

Now D/V is the initial concentration of the drug, so for the experiment described above the initial concentration was 12 mg/l. This relation would normally be used in the form $D = 12V$ to estimate the required dose for a patient for whom V is known, when an initial drug concentration of 12 mg/l (in the middle of the therapeutic range) is desired. It is found in practice that there is a simple empirical relationship between the apparent volume of distribution V (in litres) and the patient's weight W (in kilograms), namely

$$V = \tfrac{1}{2}W.$$

This enables one to estimate the correct dose directly from the patient's weight.

Quantitative solution to the problem

When a patient suffering from a bad asthma attack is admitted to hospital, he or she must be put quickly onto a routine of drug therapy which will keep theophylline in the bloodstream at a concentration lying within the therapeutic range, say between 12 mg/l and 6 mg/l, to be on the safe side. We shall devise a system of treatment which will serve this purpose. This requires the following three steps.

1. Find what dose is required to produce an initial concentration of the drug of 12 mg/l.
2. Find how long it will be before the concentration falls to 6 mg/l, when a further injection will be necessary.
3. Find what dose is required to bring the concentration up to 12 mg/l again.

Further injections of the dose found in step 3 may then be repeated at the intervals found in step 2 for so long as the treatment is required.

Suppose that the patient weighs W kg; then the apparent volume of distribution is $\frac{1}{2}W$ litres, so in order to achieve an initial drug concentration of 12 mg/l we would need to inject the patient with $12 \times \frac{1}{2}W = 6W$ mg of the drug.

Our model says that

$$c = 12e^{-0.17t},$$

so that to find when the value of c drops to 6 we must solve the equation

$$6 = 12e^{-0.17t}$$

for t. The solution is

$$t = \frac{\log_e 2}{0.17} \simeq \frac{0.69}{0.17} \simeq 4.$$

Thus after about 4 hours the concentration will have dropped to about 6 mg/l. To boost it up to 12 mg/l again requires just half of the initial dose, that is, $3W$ mg.

The instructions to the nursing staff might look as follows.

1. Weigh the patient on admission.
2. Give an initial dose of 60 mg for every 10 kg of the patient's weight.
3. Give a booster dose of half the initial dose every four hours.

Evaluation of the model

Reliability of the quantitative solution

We have assumed that there is a sufficient safety margin in working well within the therapeutic range that we need consider the patient's weight only to the nearest 10 kilograms. It is certainly a good idea not to complicate nursing procedures unduly, by demanding that the dose be calculated correct to ten places of decimals, or that the period between injections be timed to the nearest second. Nursing procedures must be pretty well-conditioned to small changes in the data to be practical.

Shortcomings of the model

There is bound to be uncertainty in any model of a bodily function, and in particular the value of the constant λ will vary somewhat from patient to patient. Moreover, the relation between apparent volume of distribution and body weight is not a precise law. It is unrealistic to expect that the doses will be repeated at exactly four-hourly intervals, however well run the hospital is. Indeed, the therapeutic limits themselves may differ between individuals.

Another consideration relates to assumption (b), that the drug is instantaneously mixed uniformly through the bloodstream. All the concentrations calculated in the model depend on this assumption. Now immediately after injection the drug is not uniformly distributed but is concentrated in a small region of the bloodstream; and so initially the true concentration, in this region, is much higher than the calculated one. It follows that caution is necessary if toxicity is to be avoided.

The kind of treatment described above is feasible, but it requires a lot of injections. From the patient's point of view this is the most serious defect of the model, and it would be better to devise a method of administering the drug which avoided repeated injections.

Revisions to the model

Several possibilities for improving the model suggest themselves. One is to find out how sensitive the conclusions drawn from the model are to variations in the value of λ, the relation between V and W, the measurement of W, and the time between injections. Another would be to model the mixing process which takes place after the injection while the drug is being distributed through the bloodstream. However, the most important consideration so far as the patient is concerned is to devise a better way of administering the drug. This leads to the following revised model.

Revised model

The alternative strategy is to give the patient a slow intravenous infusion after the initial injection. This continuously replenishes the supply of drug in the blood. The model has to be changed if continuous infusion is considered. In particular, assumption (d) must be revised, which leads to an altered differential equation.

The revised model will consist of a differential equation for the quantity of drug in the body as a function of time, as for the intermittent case. A new symbol must be introduced for the rate of infusion: suppose it to be I mg/hour.

The input-output principle may be applied; the version given previously still applies to the excretion of the drug by the kidneys, but now the drug is being continuously replenished so we must incorporate an input term, which is $I\delta t$. Thus

$$q(t+\delta t) - q(t) \simeq I\delta t - \lambda q(t)\delta t,$$

whence

$$\frac{dq}{dt} = I - \lambda q.$$

Solution of the revised problem

The general solution of this differential equation is

$$q(t) = I/\lambda + ke^{-\lambda t}$$

for some constant k. But note that the differential equation has a constant solution $q = I/\lambda$; ideally, one would like to choose I so that the value of I/λ corresponds to a concentration which lies in the middle of the therapeutic range, since a suitable initial dose will then ensure that the concentration remains always at this level. We should therefore take (say)

$$I/\lambda = 12;$$

but $\lambda = 0.17$, so this requires that

$$I = 2.04,$$

and the drug should be infused at the rate of about 2 mg/hour.

Evaluation of the revised model

In fact, it follows from the general solution of the differential equation

$$q(t) = I/\lambda + ke^{-\lambda t}$$

that $q(t)$ tends to I/λ in the long run; so this is a very good regime because it tends to settle down to the desired behaviour of its own accord, and the initial dose does not have to be very accurate so long as it results in a drug concentration in the therapeutic range.

Conclusions

Conclusions

We have developed a simple exponential decay model of the concentration of a drug in the bloodstream. Using it, we have been able to predict the frequency with which doses of aminophylline must be injected into a patient with asthma in order to keep the concentration of theophylline within the therapeutic range. We have formulated the results of the investigation in a form suitable for use by hospital nursing staff on the wards. The model has been modified to cover the case of continuous intravenous infusion rather than repeated injection; this method of administering the drug has advantages for the patient's comfort, and is also very stable in the sense that the concentration tends to settle down to the required level regardless of how accurately the initial dose is measured.

Sources of data

The data for this report were kindly provided by Dr J. Aronson of the Department of Clinical Pharmacology, Oxford University.

Appendix: Solutions to the exercises

1. Here are some possible features.

- the ages of the pupils
- the nature of the subjects taught
- the type of lesson (classroom, laboratory, etc.)
- the time of day
- whether the pupils have to change classrooms
- the length of the school day
- the number of different subjects taught

2. Here are possible ratings for the nine features; you may disagree with some of them.

(i) 4 (driver's age will not affect the cost of the damage)

(ii) 1 (a head-on collision will probably result in more damage than will a glancing blow)

(iii) 4 (it is how the cars collided that matters, not why)

(iv) 2 (valuable cars are more expensive to repair)

(v) 1 (a Rolls-Royce costs more to repair than a Metro)

(vi) 4 (again, it is how the cars collided, not why, that matters)

(vii) 1 (heavy cars have more momentum, and so cause more damage in a crash)

(viii) 1 (a high speed means a greater momentum)

(ix) 3 (this may affect how the collision occurred, since skids are more likely in bad weather for example, but information about the weather by itself is not very useful)

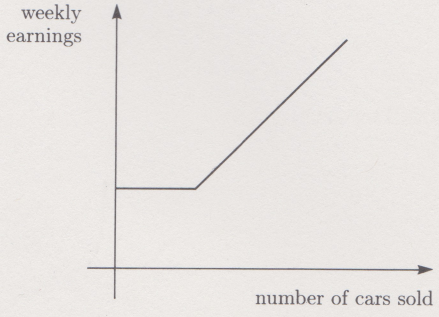
3. **(i)** The engineer must state:

the length of time for which the green, amber and red lights are 'on', for each of the two sets of lights;

the way in which the two sets of lights (one for each road) 'match up' in their timing to ensure safe crossing of the intersection.

(ii) The most important feature of the traffic to be taken into account when determining the timing of the traffic lights is the frequency of arrival of vehicles at the intersection. The engineer will need to know how many vehicles arrive per hour (say) on each road in each direction, and will have to take into account how this frequency varies during the day, and from day to day.

4.



5. Graph C

6. **(i)** Here is an initial feature list.

- speed of projection of the shot
- strength of the putter's arm
- one-handed throw or two-handed
- size of the shot-putter
- mass of the shot-putter
- type of motion of the putter (i.e. smooth or 'hopping' or 'turning')
- material of the shot
- angle of projection of the shot
- air resistance on the shot
- height above the ground of the shot at release
- size of the shot
- spin of the shot
- weather conditions (wind, humidity, temperature)
- mass of the shot
- gravity
- speed of the shot-putter across the putting circle

Your list may be different, but should overlap this one quite extensively.

(ii) The following list is a refined list of the features affecting the motion of the shot through the air. It has been subdivided into two groups, the first consisting of features affecting the shot while it is in motion, the second initial conditions.

Shot in the air

- air resistance on the shot
- size of the shot
- spin of the shot
- weather conditions
- mass of the shot
- gravity

Initial conditions

- speed of projection
- angle of projection
- height above ground

Here are the features affecting the shot-putter's action.

- strength of putter's arm
- one-handed throw or two-handed
- size of putter
- mass of putter
- type of motion
- speed of putter

There does not seem to be much advantage in grouping them further.

(iii) The essential features of the motion of the shot, which can be incorporated in a simple model, are as follows.

Shot in the air

- mass of the shot
- gravity

Initial conditions

- speed of projection
- angle of projection
- height above ground

Here is a corresponding pruned list for the shot-putter's action.

- strength of putter's arm
- size of putter
- mass of putter
- speed of putter

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